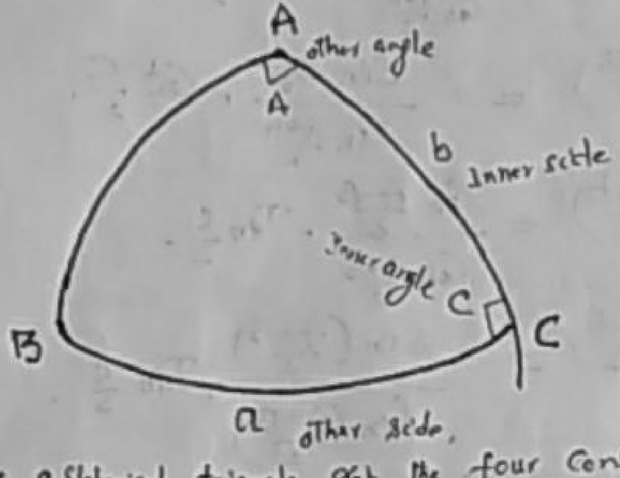
Cotangent Formula

(Relation between four Consecutive elements of a Spherical Triangle)



Let ABC be a spherical triangle. Let the four consecutive elements be taken as

a, C, b, A

The side b which is between two angles A and C is termed as inner side. & side a as other side.

Similarly the angle C is between two sides a & b is termed as inner angle and the angle A as other angle.

We have from ΔABC

$$\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos A \quad \text{———} \quad (1)$$

$$\cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos C \quad \text{———} \quad (2)$$

Also from Sine formula, we have

$$\frac{\sin a}{\sin A} = \frac{\sin c}{\sin C}$$

$$\therefore \sin c = \sin C \cdot \frac{\sin a}{\sin A} \quad \text{———} \quad (3)$$

Putting the values of $\cos c$ & $\sin c$ from (2) & (3) in (1), we get

$$\cos a = \cos b(\cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos C) + \sin b \cdot \sin C \cdot \frac{\sin a}{\sin A} \cdot \cos A$$

$$= \cos a \cos^2 b + \sin a \cdot \sin b \cdot \cos b \cdot \cos C + \sin a \cdot \sin b \cdot \sin C \cdot \cot A$$

$$\Rightarrow \cos a (1 - \cos^2 b) = \sin a \cdot \sin b \cdot \cos b \cdot \cos C + \sin a \cdot \sin b \cdot \sin C \cdot \cot A$$

$$\Rightarrow \cos a \cdot \sin^{\cancel{2}} b = \sin a \cdot \cancel{\sin b} \cdot (\cos b \cdot \cos C + \sin C \cdot \cot A) \quad *$$

$$* \Rightarrow \cot a \cdot \sin b = \cos b \cdot \cos C + \sin C \cdot \cot A$$

$$\Rightarrow \cos b \cdot \cos C = \sin b \cdot \cot a - \sin C \cdot \cot A$$

$$\Rightarrow \cos(\text{Inner side}) \cdot \cos(\text{Inner angle}) = \sin(\text{inner side}) \cdot \cot(\text{other side}) - \sin(\text{inner angle}) \cdot \cot(\text{other angle})$$

Napier's Analogies

In a spherical Triangle ABC, prove that

(i) $\tan\left(\frac{A+B}{2}\right) = \frac{\cos\frac{a-b}{2}}{\cos\frac{a+b}{2}} \cdot \cot\frac{C}{2}$.

(ii) $\tan\left(\frac{A-B}{2}\right) = \frac{\sin\frac{a-b}{2}}{\sin\frac{a+b}{2}} \cdot \cot\frac{C}{2}$.

(iii) $\tan\left(\frac{a+b}{2}\right) = \frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}} \cdot \tan\frac{c}{2}$.

(iv) $\tan\left(\frac{a-b}{2}\right) = \frac{\sin\left(\frac{A-B}{2}\right)}{\sin\left(\frac{A+B}{2}\right)} \cdot \tan\frac{c}{2}$.

Proof of (i) By sine formula, we have

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = K \text{ (say)} \quad \text{——(1)}$$

$$\therefore \sin A + \sin B = K(\sin a + \sin b) \quad \text{——(2)}$$

Again by supplemental cosine formula, we have

$$\cos A = -\cos B \cos C + \sin B \sin C \cdot \cos a$$

$$\cos B = -\cos C \cdot \cos A + \sin C \cdot \sin A \cdot \cos b$$

$$\therefore \cos A + \cos B = -\cos C(\cos A + \cos B) + \sin C(\sin B \cdot \cos a + \sin A \cdot \cos b)$$

$$\Rightarrow (\cos A + \cos B)(1 + \cos C) = \sin C(K \sin b \cdot \cos a + K \sin a \cdot \cos b)$$

$$= K \sin C \cdot \sin(a+b)$$

$$\therefore \cos A + \cos B = \frac{K \sin C \cdot \sin(a+b)}{1 + \cos C} = \frac{K \cdot \cancel{2} \sin\frac{C}{2} \cdot \cos\frac{C}{2} \cdot \sin(a+b)}{\cancel{2} \cos^2\frac{C}{2}}$$

$$= K \cdot \tan\frac{C}{2} \cdot \sin(a+b) \quad \text{——(3)}$$

Dividing (2) by (3), we have

$$\frac{\sin A + \sin B}{\cos A + \cos B} = \frac{\cancel{K}(\sin a + \sin b)}{\cancel{K} \tan\frac{C}{2} \cdot \sin(a+b)}$$

$$\Rightarrow \frac{\cancel{2}\sin\frac{A+B}{2}\cdot\cos\frac{A-B}{2}}{\cancel{2}\cos\frac{A+B}{2}\cdot\cancel{\cos\frac{A-B}{2}}} = \frac{\cancel{2}\sin\frac{a+b}{2}\cdot\cos\frac{a-b}{2}}{\tan\frac{C}{2}\cdot\cancel{2}\sin\frac{a+b}{2}\cdot\cos\frac{a+b}{2}}$$

$$\Rightarrow \boxed{\tan\frac{A+B}{2} = \frac{\cos\left(\frac{a-b}{2}\right)}{\cos\left(\frac{a+b}{2}\right)}\cdot\cot\frac{C}{2}.}$$

Proved.

Proof of (ii)

From sine formula $\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} = k$ (say)

$\therefore \sin A - \sin B = K(\sin a - \sin b)$ ——— (1)

Again by supplemental Cosine formula, we have

$\cos A = -\cos B\cdot\cos C + \sin B\cdot\sin C\cdot\cos a$

$\cos B = -\cos C\cdot\cos A + \sin A\cdot\sin C\cdot\cos b$

$\therefore \cos A+\cos B = -\cos C(\cos A + \cos B) + \sin C(\sin B\cdot\cos a + \sin A\cdot\cos b)$

$\Rightarrow (\cos A+\cos B)(1+\cos C) = \sin C(K\sin b\cdot\cos a + K\sin a\cdot\cos b)$

$= K\sin C\cdot\sin(a+b)$

$\therefore \cos A+\cos B = \frac{K\sin C\cdot\sin(a+b)}{1+\cos C} = \frac{K\cancel{2}\sin\frac{C}{2}\cdot\cos\frac{C}{2}}{\cancel{2}\cos^2\frac{C}{2}}\cdot\sin(a+b)$

$= K\cdot\tan\frac{C}{2}\cdot\sin(a+b)$ ——— (2)

Dividing (1) by (2), we have

$$\frac{\sin A-\sin B}{\cos A+\cos B} = \frac{K\cdot\cancel{2}\cos\frac{a+b}{2}\cdot\sin\frac{a-b}{2}}{K\cdot\tan\frac{C}{2}\cdot\cancel{2}\sin\frac{a+b}{2}\cdot\cos\frac{a+b}{2}}$$

$$\Rightarrow \frac{\cancel{2}\cos\frac{A+B}{2}\cdot\sin\frac{A-B}{2}}{\cancel{2}\cos\frac{A+B}{2}\cdot\cos\frac{A-B}{2}} = \frac{\sin\frac{a-b}{2}}{\sin\frac{a+b}{2}}\cdot\cot\frac{C}{2}$$

$$\Rightarrow \boxed{\tan\frac{A-B}{2} = \frac{\sin\left(\frac{a-b}{2}\right)}{\sin\left(\frac{a+b}{2}\right)}\cdot\cot\frac{C}{2}}$$

Proved.

(iii) For the ~~pta~~ polar triangle $A'B'C'$, the formula

$$\tan\left(\frac{A+B}{2}\right) = \frac{\cos\left(\frac{a-b}{2}\right)}{\cos\left(\frac{a+b}{2}\right)} \cdot \cot\frac{C}{2}$$

takes the form

$$\tan\left(\frac{A'+B'}{2}\right) = \frac{\cos\left(\frac{a'-b'}{2}\right)}{\cos\left(\frac{a'+b'}{2}\right)} \cdot \cot\frac{C'}{2}$$

But $A' = \pi - a$, $B' = \pi - b$, $C' = \pi - c$

& $a' = \pi - A$, $b' = \pi - B$, $c' = \pi - C$

Hence $\tan\left(\frac{\pi - a + \pi - b}{2}\right) = \frac{\cos\left(\frac{\pi - A - \pi + B}{2}\right)}{\cos\left(\frac{\pi - A + \pi - B}{2}\right)} \cdot \cot\left(\frac{\pi - c}{2}\right)$

$\Rightarrow \tan\left(\pi - \frac{a+b}{2}\right) = \frac{\cos\left(\frac{B-A}{2}\right)}{\cos\left(\frac{2\pi - A - B}{2}\right)} \cdot \cot\left(\frac{\pi}{2} - \frac{c}{2}\right)$

$\Rightarrow -\tan\left(\frac{a+b}{2}\right) = \frac{\cos\left(\frac{A-B}{2}\right)}{\cos\left(\pi - \frac{A+B}{2}\right)} \cdot \tan\frac{c}{2}$

$\Rightarrow -\tan\left(\frac{a+b}{2}\right) = \frac{\cos\left(\frac{A-B}{2}\right)}{-\cos\left(\frac{A+B}{2}\right)} \cdot \tan\frac{c}{2}$

$\Rightarrow \tan\left(\frac{a+b}{2}\right) = \frac{\cos\left(\frac{A-B}{2}\right)}{\cos\left(\frac{A+B}{2}\right)} \cdot \tan\frac{c}{2}$

(iv) For the polar triangle $A'B'C'$, the formula $\tan\left(\frac{A-B}{2}\right) = \frac{\sin\left(\frac{a-b}{2}\right)}{\sin\left(\frac{a+b}{2}\right)} \cdot \cot\frac{C}{2}$ takes the form $\tan\left(\frac{A'-B'}{2}\right) = \frac{\sin\left(\frac{a'-b'}{2}\right)}{\sin\left(\frac{a'+b'}{2}\right)} \cdot \cot\frac{C'}{2}$

But $A' = \pi - a$, $B' = \pi - b$, $C' = \pi - c$

& $a' = \pi - A$, $b' = \pi - B$, & $c' = \pi - C$

Hence $\tan\left(\frac{\cancel{\pi} - a - \cancel{\pi} + b}{2}\right) = \frac{\sin\left(\frac{\pi - A - \pi + B}{2}\right)}{\sin\left(\frac{\pi - A + \pi - B}{2}\right)} \cdot \cot\left(\frac{\pi - c}{2}\right)$

$\Rightarrow -\tan\left(\frac{a-b}{2}\right) = \frac{-\sin\left(\frac{A-B}{2}\right)}{\sin\left(\pi - \frac{A+B}{2}\right)} \cdot \tan c/2$

* $\Rightarrow \tan\left(\frac{a-b}{2}\right) = \frac{\sin\left(\frac{A-B}{2}\right)}{\sin\left(\frac{A+B}{2}\right)} \cdot \tan c/2$ Proved